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SUPERADDITIVITY FOR SOLUTIONS OF COALITIONAL GAMES

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Abstract

It is well-known that the Shapley value is an additive and even linear solution concept. In this paper we describe a suitable basis for the null space of the Shapley value and give an algorithm to compute the Shapley value in a surveyable manner. The core is known to be a superadditive solution. We prove that it is additive for convex games. In terms of a new concept—maximal tightly balanced collection—we describe the dimension of the core and cones of games where the least core is superadditive.

1. Introduction

The Shapley value and the core are two major solution concepts in the theory of cooperative games in coalitional form. For a definition and the most basical properties of these concepts we refer to the books of Owen (1982) and Ichiishi (1983).

The Shapley value is a surjective linear map of the linear vector space G^N of games with player set N to $\mathbf{R}^N$. In section 2 of this note we study the null space of this linear map i.e. the set of all games with Shapley value zero. We introduce a basis $\{w_S\}_{S\subset N,\, S\neq\emptyset}$ of G^N such that the coordinates of a game v with respect to this basis are the 'potentials' $P(S,v)$ of subgames (S,v) of v in the sense of Hart/Mas-Colell (1989). This means that for every game $v \in G^N$:

$$v = \sum_{S\subset N,\, S\neq\emptyset} P(S,v)w_S \text{ or—in other words—} P(T,w_S) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{if } T \neq S \end{cases} \quad \text{for all } S \subset N,\ S \neq \emptyset.$$

As the Shapley value of the games w_S with $1 \leq |S| \leq n-2$ is shown to be the zero vector, linear combinations of these games can be added to a game v to get a game $\bar{v}$ with $\bar{v}(S) = 0$ for all coalitions S with $|S| \leq n-2$ and the same Shapley value as v. For such games $\bar{v}$ the Shapley value is computed by means of potentials.

The core is a superadditive solution concept. In section 3 we prove that the core is additive when restricted to the set of convex games. Furthermore, we assign to each game $v \in BG^N$ a uniquely determined balanced collection $\mathcal{B}$ of coalitions of N and prove that the dimension of the core of a game and the rank of the collection $\mathcal{B}$ assigned to v add up to $n = |N|$. In section 4 we study the least core of a game and give maximal subsets of G^N on which the least core is superadditive.

Let us recall some of the concepts to be used in this paper. If (N, v) is a coalitional game and S is a coalition in N, then the restriction of v to the set of subcoalitions of S is called the subgame (S, v). The *imputation set* and the *core* of a game (N, v) are defined by

$$I(N,v) := \{x \in \mathbf{R}^N \mid x(N) = v(N) \quad \text{and} \quad x_i \geq v(i) \text{ for all } i \in N\}$$

and

$$C(N,v) := \{x \in \mathbf{R}^N \mid x(N) = v(N) \quad \text{and } x(S) \geq v(S) \text{ for all } S \subset N\}.$$

where $x(S) := \sum_{i \in S} x_i$. Core or imputation set of a game may be empty. We denote by IG^N the set of games in G^N with non-empty imputation set and by BG^N the set of games with non-empty core.

A game (N, v) is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all coalitions S and T. The set of all convex games is denoted by CG^N. It is well-known that $CG^N \subset BG^N \subset IG^N \subset G^N$.

2. The Shapley value.

The Shapley value ϕ is a surjective linear map from G^N to $\mathbf{R}^N$. Therefore, the null space of ϕ i.e. $\{v \in G^N \mid \phi(v) = 0\}$ is a linear subspace of G^N of dimension $2^n - n - 1$, $n = |N|$. In this section we give a basis of G^N which lies nicely with respect to the null space of ϕ. The result is used to derive an algorithm for computing the Shapley value.

Two bases of G^N are widely used in cooperative game theory.

(1) The canonical basis $\{\delta_S\}_{S \subset N, S \neq \emptyset}$, defined by

$$\delta_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{else} \end{cases}$$

The coordinates of a game v with respect to the canonical basis are the values $v(S)$ of the coalitions S. The Shapley value has a rather complicated expression in these coordinates (Shapley (1953)):

$$\phi(v)_i = \sum_{S \subset N \setminus \{i\}} \gamma_s [v(S \cup \{i\}) - v(S)] \text{ where } \gamma_s = \left(n \binom{n-1}{s}\right)^{-1} \text{ and } s = |S|.$$

The computation of the Shapley value by using this formula requires

$$\sum_{s=0}^{n-1} (n-s) \binom{n}{s} = n \cdot 2^{n-1}$$

computational steps, for there are $\binom{n}{s}$ coalitions with s elements and for each of these coalitions there are $n - s$ possible choices of $i \notin S$. The computational complexity of an algorithm based on this formula of the Shapley value is $O(m \log m)$ where $m = 2^n$, the size of the input.

(2) Another frequently used basis of G^N is formed by the unanimity games $\{u_S\}_{S\subset N,\, S\neq\emptyset}$ defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S\subset T \\ 0 & \text{else} \end{cases}$$

The coordinates of a game with respect to the basis of unanimity games $\{\Delta(S)\}_{S\subset N,\, S\neq\emptyset}$ are the 'dividends' of coalitions S in the game v (cf. Harsanyi (1959) and Maschler (1982)). The Shapley value of a game v is an easier expression in terms of the 'dividends'.

$$\phi(v)_i = \sum_{T:\, i\in T} |T|^{-1}\Delta(T).$$

Although this formula is simpler, an algorithm for the Shapley value based on this formula requires

$$\sum_{s=1}^{n}\binom{n}{s}(2^{n-s}+s) = 3^n - 2^n + n\cdot 2^{n-1}$$

computational steps (for a coalition S with s players we have to change the value of all coalitions containing S and to divide the dividend among the players of S) and it has, therefore, the complexity $O(m^{\log 3})$ (we mean the 2-logarithm).

Recently, Hart/ Mas-Colell (1989) introduced the so-called 'potentials' of subgames (S, v). They were defined recursively:

$$P(\{i\}, v) := v(i) \text{ for all } i\in N \text{ and } P(S, v) := |S|^{-1}[v(S) + \sum_{i\in S} P(S\backslash\{i\}, v)] \text{ if } |S|\geq 2. \tag{2.1}$$

In terms of the potentials the Shapley value has a very simple form

$$\phi(v)_i = P(N, v) - P(N\backslash\{i\}) \text{ for all } i\in N. \tag{2.2}$$

From (2.1) follows easily that the potentials are linear functions on G^N and that only the zero game has all potentials equal to zero. This means that the potentials of a game are coordinates with respect to a basis of G^N. We describe this basis in the next theorem.

Theorem 1. *For every coalition $S\subset N$, $S\neq\emptyset$ there is exactly one game w_S with the property that $P(T, w_S) = 1$ if $T = S$ and $P(T, w_S) = 0$ for all coalitions $T\neq S$. Moreover, w_S is given by*

$$w_S(T) = \begin{cases} |S| & \text{if } T = S \\ -1 & \text{if } T = S\cup\{j\} \text{ with } j\notin S\,. \\ 0 & \text{else} \end{cases} \tag{2.3}$$

Proof. There is exactly one game w_S because the linear map which assigns to a game v the potentials $\{P(S, v)\}_{S\subset N,\, S\neq\emptyset}$ is a linear isomorphism. We use the formula (2.1) to find the game w_S. Let $T\subset N$ such that $S\not\subset T$ or $|T|\geq |S|+2$. Then $P(T, w_S) = 0$ and $P(T\backslash\{i\}, w_S) = 0$ for all $i\in T$. From (2.1) we infer that $w_S(T) = 0$. For $T = S$ formula (2.1) gives $P(S, w_S) = |S|^{-1}w_S(S)$ i.e. $w_S(S) = |S|$. If $T = S\cup\{j\}$ with $j\notin S$, then we find $0 = P(S\cup\{j\}, w_S) = (|S|+1)^{-1}[w_S(S\cup\{j\}) + P(S, w_S)]$. Therefore $w_S(S\cup\{j\}) = -1$. The result follows. QED

Corollary. *The games w_S with $1 \leq |S| \leq n-2$ together with the game $w := w_N + \sum_{i \in N} w_{N \backslash \{i\}}$ span the null space of ϕ.*

Proof. The set of games $\{w_S\}_{1 \leq |S| \leq n-2}$ span the subspace of games v with $P(N, v) = P(N\backslash\{i\}, v) = 0$ for all $i \in N$. The game w has $P(N, w) = P(N\backslash\{i\}, w) = 1$ for all $i \in N$. From (2.2) we infer that the Shapley value of all these games is zero. The games are clearly linearly independent and their number is $2^n - n - 1$, the dimension of the null space of ϕ. QED

Algorithm. We can use the Corollary to compute the Shapley value of a game v in a surveyable manner. If we subtract from the game v linear expressions in the games $\{w_S\}_{1 \leq |S| \leq n-2}$, then we get games with the same Shapley value as the game v. By choosing the linear expressions properly, the final game $\bar{v}$ will have $\bar{v}(S) = 0$ for all coalitions S with $|S| \leq n-2$. Then we compute the potentials $P(N, \bar{v})$ and $P(N\backslash\{i\}, \bar{v})$ for all $i \in N$ and the Shapley value of $\bar{v}$ from (2.2). In an algorithmic form this looks as follows.

step 1 **while** there is a coalition S with $v(S) \neq 0$ and $1 \leq |S| \leq n-2$
  begin: Choose $S \subset N$ with $v(S) \neq 0$, $1 \leq |S| \leq n-2$ and
    $v(T) = 0$ for all $T \subset S$, $T \neq S$.
    for all $j \notin S$: $v(S \cup \{j\}) = v(S \cup \{j\}) + |S|^{-1} v(S)$: **next** j
    $v(S) = 0$
  end
repeat

step 2 **for all** $i \in N$: $x_i = (n-1)^{-1} v(N\backslash\{i\})$: **next** i
  $t = n^{-1}[v(N) + \sum_{i \in N} x_i]$
  for all $i \in N$: $\phi(v)_i = t - x_i$: **next** i

This algorithm requires

$$\sum_{s=1}^{n-2} \binom{n}{s} (n - s + 1) + 2n + 1 = 2^n + n \cdot 2^{n-1} - n - 1$$

computational steps (for a coalition S with s players $(1 \leq s \leq n-2)$ we have to change the value of $n - s + 1$ coalitions and step 2 requires $2n + 1$ steps) and has, therefore, the complexity $O(m \log m)$ where $m = 2^n$. So, the complexity of this algorithm is lower than that of the Harsanyi method and the algorithm is—in our opinion—more surveyable than the algorithm based on Shapley's formula. We demonstrate the computation in the following example.

S	(1)	(2)	(3)	(4)	(12)	(13)	(14)	(23)	(24)	(34)	(123)	(124)	(134)	(234)	N
$v(S)$	7	3	-1	-2	5	8	9	6	4	9	8	10	20	15	21
	0	0	0	0	12 15 0	15 14 0	16 14 0	9 8 0	7 5 0	8 6 0	15.5 22.5 26.5	17.5 24.5 27	27 34 37	19 21.5 24.5	
$\bar{v}(S)$	0	0	0	0	0	0	0	0	0	0	26.5	27	37	24.5	21

$$x_1 = 8\tfrac{1}{6},\ x_2 = 12\tfrac{1}{3},\ x_3 = 9,\ x_4 = 8\tfrac{5}{6} \text{ en } t = 14\tfrac{5}{6}$$
$$\phi(v)_1 = 6\tfrac{2}{3},\ \phi(v)_2 = 2\tfrac{1}{2},\ \phi(v)_3 = 5\tfrac{5}{6},\ \phi(v)_4 = 6.$$

3. The core.

In Peleg (1986) superadditivity (SUPA) is one of the characterizing properties of the core i.e. if $v, w \in BG^N$, then $C(v) + C(w) \subset C(v + w)$. In the following proposition we prove that the core is additive on the cone CG^N of convex games.

Proposition 2. *If $v, w \in CG^N$ then $C(v) + C(w) = C(v + w)$.*

Proof. Let $\mathcal{W}(v)$ be the Weber set of a game v i.e. the convex hull of the marginal vectors $\{m^\sigma(v) \mid \sigma: \{1, \ldots, n\} \to N$ is a bijective map $\}$ where

$$m^\sigma(v)_{\sigma(i)} = v(\{\sigma(1), \ldots, \sigma(i)\}) - v(\{\sigma(1), \ldots, \sigma(i-1)\})$$
for all $i \in \{1, \ldots, n\}$ and all bijections $\sigma: \{1, \ldots, n\} \to N$.

It is well known that the Weber set and the core of a game v coincide when the game v is convex (Ichiishi (1983)). Furthermore, the Weber set is subadditive i.e. $\mathcal{W}(v) + \mathcal{W}(w) \supset \mathcal{W}(v + w)$ for all games $v, w \in G^N$. This follows from the fact that $m^\sigma(v) + m^\sigma(w) = m^\sigma(v + w)$ for all bijections σ and all games $v, w \in G^N$ and therefore, if x is a convex combination of marginal vectors of $v + w$, say $x = \sum_\sigma a^\sigma m^\sigma(v + w)$, then

$$x = \sum_\sigma a^\sigma (m^\sigma(v) + m^\sigma(w)) = \sum_\sigma a^\sigma m^\sigma(v) + \sum_\sigma a^\sigma m^\sigma(w) \in \mathcal{W}(v) + \mathcal{W}(w).$$

If $v, w \in CG^N$, then we find

$$\mathcal{W}(v + w) \subset \mathcal{W}(v) + \mathcal{W}(w) = C(v) + C(w) \subset C(v + w) = \mathcal{W}(v + w).$$

Hence, the core is additive on CG^N. QED

Let v be a game and let $\mathcal{B}$ be a non-empty collection of coalitions.

The collection $\mathcal{B}$ is *tightly balanced with respect to the game* v if there are positive real numbers $\{y_S\}_{S\in\mathcal{B}}$ such that $\sum_{S\in\mathcal{B}} y_S e_S = e_N$ and $\sum_{S\in\mathcal{B}} y_S v(S) = v(N)$.

A tightly balanced collection $\mathcal{B}$ is *maximal* if there is no strictly larger tightly balanced collection. Since the union of tightly balanced collections is tightly balanced too and $\{N\}$ is tightly balanced for every game $v \in G^N$, there is exactly one maximal tightly balanced collection to each game $v \in G^N$ and the grand coalition N is a member of this collection. Finally, the *rank* of a collection $\mathcal{B}$ is the rank of the set of vectors $\{e_S \mid S \in \mathcal{B}\}$.

The next theorem gives a relation between the dimension of the core of a game and the rank of its maximal tightly balanced collection.

Theorem 3. *Let v be a balanced game. The dimension of the core of v and the rank of the collection $\mathcal{B}$ which is maximal tightly balanced with respect to v, add up to $n = |N|$.*

Proof. Let v be a balanced game and $\mathcal{B} := \{S \subset N \mid x(S) = v(S) \text{ for all } x \in C(v)\}$.

Core elements of v are the optimal solutions of the LP-problem

$$\text{(P)} \qquad x(S) \geq v(S) \quad \text{for all } S \subset N, \quad \min x(N).$$

The dual problem has the following form:

$$\text{(D)} \qquad \sum_{S:\, i\in S} y_S = 1 \quad \text{for all } i \in N, \quad y_S \geq 0 \quad \text{for all } S \subset N, \quad \max \sum_{S\subset N} y_S v(S).$$

Every optimal solution $\hat{x}$ of (P) (core element of v) has $\hat{x}(S) = v(S)$ for all $S \in \mathcal{B}$ and therefore, the dual problem (D) has an optimal solution $\{\hat{y}_S\}_{S\subset N}$ with $\hat{y}_S > 0$ if and only if $S \in \mathcal{B}$ (strong complementary slackness, cf. Schrijver (1986)). This means that $\sum_{S\in\mathcal{B}} \hat{y}_S e_S = e_N$ and $\sum_{S\in\mathcal{B}} \hat{y}_S v(S) = \min \hat{x}(N) = v(N)$. The collection $\mathcal{B}$ is tightly balanced with respect to v.

Suppose that $\mathcal{B}'$ is tightly balanced with respect to v and x is a core element of v. Then:

$$x(N) = \sum_{S\in\mathcal{B}'} y'_S x(S) \geq \sum_{S\in\mathcal{B}'} y'_S v(S) = v(N) = x(N)$$

if $\sum_{S\in\mathcal{B}'} y'_S e_S = e_N$, $y'_S > 0$ for all $S \in \mathcal{B}'$ and $\sum_{S\in\mathcal{B}'} y'_S v(S) = v(N)$. Hence, $x(S) = v(S)$ for all $S \in \mathcal{B}'$ i.e. $\mathcal{B}' \subset \mathcal{B}$. So $\mathcal{B}$ is the maximal tightly balanced collection of v. Note that there is a point $x^0 \in C(v)$ such that $x^0(S) = v(S)$ if and only if $S \in \mathcal{B}$. Then $x \in C(v)$ implies that $x(S) = x^0(S)$ for all $S \in \mathcal{B}$ and hence, $C(v) \subset x^0 + \{y \in \mathbf{R}^N \mid (e_S, y) = 0 \text{ for all } S \in \mathcal{B}\}$. Hence, the dimension of the core is at most $n - \operatorname{rank}\mathcal{B}$. But, if $y(S) = 0$ for all $S \in \mathcal{B}$, then $x^0 + \delta y \in C(v)$ if $|\delta|$ is small enough: $\dim C(v) + \operatorname{rank} \mathcal{B} = n = |N|$. QED

For every balanced collection $\mathcal{B} \supset \{N\}$ we define

$BG^N(\mathcal{B}) := \{v \in BG^N \mid \mathcal{B}$ is the unique maximal tightly balanced collection w.r.t. $v\}$.

Then BG^N is the disjoint union of the sets $BG^N(\mathcal{B})$ where $\mathcal{B}$ runs through the balanced collections containing N.

Proposition 4. *The sets $BG^N(\mathcal{B})$ are non-empty cones in BG^N and the closure of $BG^N(\mathcal{B})$ is the union of all cones $BG^N(\mathcal{B}')$ with $\mathcal{B}' \supset \mathcal{B}$.*

Proof. Let $\mathcal{B}$ be any balanced collection containing N and let $v \in BG^N$ be the game defined by

$$v(S) = \begin{cases} 0 & \text{if } S \in \mathcal{B} \\ -1 & \text{else} \end{cases}$$

Then it is easy to show that $v \in BG^N(\mathcal{B})$. If $v, w \in BG^N(\mathcal{B})$, then there exist $x^0 \in C(v)$ and $y^0 \in C(w)$ such that $x^0(S) = v(S)$ if and only if $S \in \mathcal{B}$ and $y^0(S) = w(S)$ if and only if $S \in \mathcal{B}$. Then $x^0 + y^0 \in C(v+w)$ by the superadditivity of the core and $x^0(S) + y^0(S) = (v+w)(S)$ if and only if $S \in \mathcal{B}$. This means that $v + w \in BG^N(\mathcal{B}')$ with $\mathcal{B}' \subset \mathcal{B}$. If $z \in C(v+w)$ and $\sum_{S \in \mathcal{B}} y_S e_S = e_N$ then

$$z(N) = \sum_{S \in \mathcal{B}} y_S z(S) \geq \sum_{S \in \mathcal{B}} y_S (v+w)(S) = \sum_{S \in \mathcal{B}} y_S v(S) + \sum_{S \in \mathcal{B}} y_S w(S) = v(N) + w(N).$$

This implies that $z(S) = v(S) + w(S)$ for all $S \in \mathcal{B}$ and all $z \in C(v+w)$: $v + w \in BG^N(\mathcal{B})$. Let $v \in BG^N(\mathcal{B}')$ with $\mathcal{B}' \supset \mathcal{B}$. Define for $t > 0$ the game $v_t(S) = v(S)$ for $S \in \mathcal{B}$ and $S \notin \mathcal{B}'$ and $v_t(S) = v(S) - t$ if $S \in \mathcal{B}' \backslash \mathcal{B}$. Then $v_t \in BG^N(\mathcal{B})$ for all $t > 0$ and v_t converges to v when t goes to zero: $BG^N(\mathcal{B}') \subset \mathrm{cl}(BG^N(\mathcal{B}))$ if $\mathcal{B}' \supset \mathcal{B}$.

If $\sum_{S \in \mathcal{B}} y_S e_S = e_N$ with $y_S > 0$ for all $S \in \mathcal{B}$, then $\sum_{S \in \mathcal{B}} y_S v(S) = v(N)$ for all $v \in BG^N(\mathcal{B})$. The set of games satisfying the last equality is a closed set in BG^N and contains therefore $\mathrm{cl}(BG^N(\mathcal{B}))$ i.e. if v is a game in the closure of $BG^N(\mathcal{B})$ then $\mathcal{B}$ is tightly balanced with respect to v and $v \in BG^N(\mathcal{B}')$ for some balanced collection $\mathcal{B}' \supset \mathcal{B}$. QED

Remark. If $v \in BG^N$ and $\dim C(v) = n - 1$ then $v \in BG^N(\mathcal{B}_0)$ where $\mathcal{B}_0 = \{N\}$. If $\mathcal{B}_1 = \{S \subset N \mid S \neq \emptyset\}$ then $BG^N(\mathcal{B}_1) = A^N$, the set of additive games.

4. The least core.

For every player set N let $u_0^N \in G^N$ be the game with values $u_0^N(S) = 1$ whenever $S \neq \emptyset, N$ and $u_0^N(N) = u_0^N(\emptyset) = 0$. For every game $v \in G^N$ there is a real number $\varepsilon(v)$ such that $v - \varepsilon u_0^N \in BG^N$ if and only if $\varepsilon \geq \varepsilon(v)$. More precisely, the game $v - \varepsilon u_0^N \in BG^N$ if and only if for all solutions of the equation $\sum_{S \subset N} y_S e_S = e_N$ with $y_S \geq 0$ and $y_N = 0$ the inequality $\sum_{S \subset N} y_S v(S) - (\sum_{S \subset N} y_S)\varepsilon \leq v(N)$ holds. Define $\varepsilon(v) := \max\{(\sum_S y_S)^{-1}(\sum_S y_S v(S) - v(N)) \mid \sum_{S \subset N} y_S e_S = e_N$ with $y_S \geq$

0 for all $S \subset N$ and $y_N = 0\}$. Then $v - \varepsilon u_0^N \in BG^N$ if and only if $\varepsilon \geq \varepsilon(v)$. Note that the solution space of the equation $\sum_S y_S e_S = e_N$ with $y_S \geq 0$ is compact and that, therefore, $\varepsilon(v)$ is well-defined.

Remark. In Maschler/Peleg/Shapley (1979) the number $\varepsilon(v)$ occurs under the name $\epsilon(\Gamma)$.

The core of the game $v - \varepsilon(v)u_0^N$ is called the *least core* $\mathcal{LC}(v)$ of the game (N, v).

Before we come to the main result of this section we prove a useful lemma which characterizes the games v for which the core coincides with the least core.

Lemma 5. *For a game $v \in BG^N$ the following statements are equivalent:*

(1) The core $\mathcal{C}(v)$ has dimension $\leq n-2$,

(2) The core $\mathcal{C}(v)$ coincides with the least core $\mathcal{LC}(v)$,

(3) $v \in BG^N(\mathcal{B})$ where $\mathcal{B}$ is a balanced collection of rank ≥ 2.

Proof. (1) $\Leftrightarrow$ (3) From Theorem 3 we infer that the rank of the maximal tightly balanced collection $\mathcal{B}$ belonging to a game $v \in BG^N$ with core dimension $\leq n-2$ is at least 2 and, conversely, if $v \in BG^N(\mathcal{B})$ with rank $\mathcal{B} \geq 2$ then $\dim \mathcal{C}(v) \leq n-2$..

(2) $\Leftrightarrow$ (3) If (3) does not hold, then $v \in BG^N(\mathcal{B}_0)$ and there is an element $x^0 \in \mathcal{C}(v)$ such that $x^0(S) > v(S)$ for all $S \neq N, \emptyset$. Hence, $x^0(S) \geq v(S) + \eta$ for some $\eta > 0$ and all $S \neq N, \emptyset$. Then $\mathcal{C}(v) \neq \mathcal{LC}(v)$. If, conversely, (2) does not hold, then there is a number $\eta > 0$ such that $\mathcal{C}(v + \eta u_0^N) \neq \emptyset$ and, consequentially, $x(S) \geq v(S) + \eta > v(S)$ for all $S \neq N, \emptyset$ and all $x \in \mathcal{C}(v + \eta u_0^N)$. Therefore, $v \in BG^N(\mathcal{B}_0)$ and (3) does not hold. QED

If $v \in G^N$ and $\mathcal{B}$ is a balanced collection of coalitions which contains besides N also another coalition $S \neq N$, then we define $v \in G^N(\mathcal{B})$ if $v - \varepsilon(v)u_0^N \in BG^N(\mathcal{B})$. The next theorem gives a necessary and sufficient condition that

$$\mathcal{LC}(v) + \mathcal{LC}(w) \subset \mathcal{LC}(v + w).$$

Theorem 6. *If $v \in G^N(\mathcal{B})$ and $w \in G^N(\mathcal{B}')$ then: $\mathcal{LC}(v) + \mathcal{LC}(w) \subset \mathcal{LC}(v+w)$ if and only if $\mathcal{B} \cap \mathcal{B}'$ contains a balanced collection $\mathcal{B}'' \neq \{N\}$.*

Proof. Let $\bar{v} := v - \varepsilon(v)u_0^N$, $\bar{w} := w - \varepsilon(w)u_0^N$ and $\bar{u} := \bar{v} + \bar{w}$. Then $\mathcal{C}(\bar{v}) = \mathcal{LC}(v)$ and $\mathcal{C}(\bar{w}) = \mathcal{LC}(w)$. By the superadditivity of the core we find $\mathcal{LC}(v) + \mathcal{LC}(w) \subset \mathcal{C}(\bar{u})$.

Furthermore, there is an element x^0 of $\mathcal{LC}(v)$ such that $x^0(S) = \bar{v}(S)$ if and only if $S \in \mathcal{B}$ and an element y^0 of $\mathcal{LC}(w)$ such that $y^0(S) = \bar{w}(S)$ if and only if $S \in \mathcal{B}'$.

$\Leftarrow$ If $\mathcal{B} \cap \mathcal{B}'$ contains a non trivial balanced collection $\mathcal{B}''$, then we find for every $z \in \mathcal{C}(\bar{u})$ and every positive solution $\{y_S\}_{S\in B''}$ of $\sum_{S\in B''} y_S e_S = e_N$

$$\bar{u}(N) = z(N) = \sum_{S\in B''} y_S z(S) \geq \sum_{S\in B''} y_S \bar{u}(S) = \sum_{S\in B''} y_S(\bar{v}(S) + \bar{w}(S))$$
$$= \sum_{S\in B''} y_S(x^0(S) + y^0(S)) = x^0(N) + y^0(N) = \bar{u}(N).$$

Hence, $z(S) = \bar{u}(S)$ for all $S \in B''$ and $\dim(C(\bar{u})) \leq n-2$. Then $C(\bar{u}) = \mathcal{LC}(\bar{u}) = \mathcal{LC}(v+w)$ by Lemma 5.

$\Rightarrow$ If, conversely, $\mathcal{LC}(v) + \mathcal{LC}(w) \subset \mathcal{LC}(v+w)$ and $\overline{v+w} := v + w - \varepsilon(v+w)u_0^N$, then there is a balanced collection B'' with rank $B'' \geq 2$ such that $\overline{v+w} \in BG^N(B'')$ (by Lemma 5, (2) $\Rightarrow$ (3)). As $x^0 + y^0 \in \mathcal{LC}(v+w)$ and $x^0(S) + y^0(S) = \bar{v}(S) + \bar{w}(S)$ if and only if $S \in B \cap B'$ we have $B'' \subset B \cap B'$.

QED

In the next Corollary we apply the Theorems 3 and 6 in a special case.

Corollary. The core (least core) of a game v consists of one point if and only if $v \in BG^N(B)$ (or $v \in G^N(B)$), respectively with rank$(B) = n$. If $v \in G^N(B)$ and $w \in G^N(B')$ and $B \cap B'$ contains a balanced collection of rank n, then

$$\mathcal{LC}(v) + \mathcal{LC}(w) = \mathcal{LC}(v+w).$$

Finally we give an example of two games in the same cone $BG^N(B)$ such that the least core is not additive.

Example Let $N = \{1,2,3\}$ and $v_{\alpha\beta}$ defined by

$$v_{\alpha\beta}(S) = \begin{cases} 0 & \text{if } S = N, \{12\}, \{3\} \\ -\alpha & \text{if } S = \{13\}, \{23\} \\ -\beta & \text{if } S = \{1\}, \{2\} \end{cases}$$

For $\alpha, \beta > 0$ is $v_{\alpha\beta} \in BG^N(B)$ where $B = \{\{12\}, \{3\}\}$. The core(which is also the least core) of $v_{\alpha\beta}$ consists of the points $(t, -t, 0)$ with $|t| \leq \min(\alpha, \beta)$. If we choose for (α, β) the pairs (1,1/2) and (1/3,1) and call the corresponding games v and w, respectively, then

$$\mathcal{LC}(v) + \mathcal{LC}(w) = [-1/2, 1/2] + [-1/3, 1/3] = [-5/6, 5/6] \neq [-4/3, 4/3] = \mathcal{LC}(v+w).$$

References

Harsanyi, J.C. (1959). "A bargaining model for the cooperative n-person game", in: Contributions to the theory of games IV (Eds. A.W. Tucker and R.D. Luce),Princeton University Press, Princeton New Jersey, 325–355.

Hart, S. and A. Mas-Colell (1989). "Potential, value and consistency", *Econometrica* 57, 589–614.

Ichiishi, T. (1983). *Game theory for economic analysis*. Academic Press, New York.

Maschler, M. (1982). "The worth of a cooperative enterprise to each member", in: Games, Economic dynamics and Time series analysis, Physica Verlag, Wien, 67–73.

Maschler, M., B. Peleg and L.S. Shapley (1979). "Geometric properties of the kernel, nucleolus and related solution concepts", *Math. O.R.* 4, 303–338.

Owen, G. (1982) *Game theory*. (2nd edition), Academic Press, New York.

Peleg, B. (1986). "On the reduced game property and its converse", *Intern. J. Game Theory* 15, 187–200.

Schrijver, A. (1986). *Theory of linear and integer programming*. Wiley, New York, chapter 7.

Shapley, L.S. (1953). "A value for n-person games", in: Contributions to the theory of games II (Eds. H.W. Kuhn and A.W. Tucker), Princeton University Press, Princeton, New Jersey, 307–317.